Report on Centralizing Monoids on a Three-Element Set (Clone Theory and Discrete Mathematics・Algebra and Logic Related to Computer Science)

Machida, Hajime; Rosenberg, Ivo G.

数理解析研究所講究録
第 1846 巻 2013 年 53-65

Report on Centralizing Monoids on a Three-Element Set

Hajime Machida * Ivo G. Rosenberg †

Abstract

For a set A with $|A|>1$, a centralizing monoid on A is a set of unary functions defined on A which commute with all members of some set of multi-variable functions on A. In this paper we restrict ourselves to the case where A is a three-element set and present the list of all centralizing monoids on A. There are 192 centralizing monoids on a three-element set, which are divided into 48 conjugate classes.

Keywords: clone; centralizer; centralizing monoid

1 Introduction

Let A be a set with $|A|>1$. For $n>0$ denote by $\mathcal{O}_A^{(n)}$ the set of all n-variable functions defined over A having the range A, that is, the set of maps from A^n into A. Let $\mathcal{O}_A$ be the set of all functions defined over A, i.e., $\mathcal{O}_A=\bigcup_{n=1}^{\infty}\mathcal{O}_A^{(n)}$. The notion of commutation for multi-variable functions is defined as a natural generalization of commutation for unary functions. The centralizer F^* for a subset F of $\mathcal{O}_A$ is the set of functions which commute with all functions in F. A centralizing monoid is the unary part of some centralizer. For more than thirty years centralizers and centralizing monoids have been studied under various names (e.g., [Da79], [Sza85]). For our previous works on centralizing monoids refer to [MR09], [MR10] and [MR11].

The purpose of this paper is to present the list of all centralizing monoids on a three-element set. There exist 192 centralizing monoids on a three-element set, which are divided into 48 conjugate classes.

2 Definitions and Basic Facts

For functions $f\in\mathcal{O}_A^{(n)}$ and $g\in\mathcal{O}_A^{(m)}$, we say that f *commutes* with g, or f and g *commute*, if

$$f(g({}^t\boldsymbol{c}_1),\ldots,g({}^t\boldsymbol{c}_n)) \quad = \quad g(f(\boldsymbol{r}_1),\ldots,f(\boldsymbol{r}_m))$$

holds for every $m\times n$ matrix M over A with rows $\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m$ and columns $\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n$.

*Graduate School of Arts and Sciences, International Christian University, 3-10-2 Osawa, Mitaka, Tokyo 185-8585 Japan ⟨machida@math.hit-u.ac.jp⟩
†Département de mathématiques et de statistique, Université de Montréal, C.P.6128, Succ. "Centre-ville", Montréal, Québec, H3C 3J7, Canada ⟨rosenb@DMS.UMontreal.CA⟩

For example, $f \in \mathcal{O}_A^{(2)}$ and $g \in \mathcal{O}_A^{(3)}$ commute if

$$f\,(g(x_1, x_2, x_3),\, g(y_1, y_2, y_3)) \;\; = \;\; g\,(f\,(x_1, y_1),\, f\,(x_2, y_2),\, f\,(x_3, y_3))$$

holds for all $x_1, x_2, x_3,\, y_1, y_2, y_3 \in A$.

We write $f \perp g$ when f commutes with g. The binary relation $\perp$ on $\mathcal{O}_A$ is obviously a symmetric relation.

Remark

(1) For unary functions $f, g \in \mathcal{O}_A^{(1)}$, f and g commute if $f(g(x)) = g(f(x))$ holds for every $x \in A$. Thus, the commutation defined above is a natural generalization of the ordinary commutation for unary functions.

(2) For an algebra $\mathcal{A} = (A;\, F)$ and $g \in \mathcal{O}_A^{(1)}$, g is an *endomorphism* of $\mathcal{A}$ if

$$f(g(x_1), \ldots, g(x_n)) \;\; = \;\; g(f(x_1, \ldots, x_n))$$

holds for all $f \in F$ and $(x_1, \ldots, x_n) \in A^n$. This equation is equivalent to saying that f commutes with g in our terminology. Hence g is an endomorphism of $\mathcal{A}$ if and only if $f \perp g$ holds for every $f \in F$. Denote by $\mathrm{End}\,(\mathcal{A})$ the set of endomorphisms of $\mathcal{A}$, i.e., $\mathrm{End}\,(\mathcal{A}) = \{g \in \mathcal{O}_A^{(1)} \mid f \perp g \text{ for } \forall f \in F\}$.

Definition 2.1 *For $F \subseteq \mathcal{O}_A$ the* centralizer F^* *of F is defined by*

$$F^* \; = \; \{\, g \in \mathcal{O}_A \;\mid\; g \perp f \;\text{ for all }\; f \in F \,\}.$$

For any subset $F \subseteq \mathcal{O}_A$ the centralizer F^* is a clone, that is, F^* contains all projections and is closed under composition. (Note: A projection $e_i^n \in \mathcal{O}_A^{(n)}$, $1 \leq i \leq n$, is an n-variable function which always takes the i-th argument as its value.) When $F = \{f\}$ we often write f^* instead of F^*. We also write F^{**} for $(F^*)^*$. The map $F \longmapsto F^{**}$ is a closure operator on $\mathcal{O}_A$.

A non-empty subset M of $\mathcal{O}_A^{(1)}$ is a *monoid* if it is closed under composition and contains the identity *id*. The set $\mathcal{O}_A^{(1)}$ is the largest monoid on A and the singleton $\{id\}$ is the smallest monoid on A. For any centralizer F^* the unary part of F^*, i.e., $F^* \cap \mathcal{O}_A^{(1)}$, is a monoid.

We give the definition of a centralizing monoid with its equivalent properties.

Definition 2.2 *For $M \subseteq \mathcal{O}_A^{(1)}$, M is a* centralizing monoid *if M satisfies the equation*

$$M \;\; = \;\; M^{**} \cap \mathcal{O}_A^{(1)}.$$

Lemma 2.1 *For $M \subseteq \mathcal{O}_A^{(1)}$ the following conditions are equivalent.*

(1) *M is a centralizing monoid.*

(2) *For some subset $F \subseteq \mathcal{O}_A$, $M \; = \; F^* \cap \mathcal{O}_A^{(1)}$*

(3) *For some algebra $\mathcal{A} = (A;\, F)$, $M \; = \; \mathrm{End}\,(\mathcal{A})$*

The proof is straightforward. Note that Lemma 2.1 (2) asserts that a centralizing monoid is the unary part of some centralizer.

The following lemma, which we call the *Witness Lemma*, is equivalent to Lemma 2.1 (2).

Lemma 2.2 *For a monoid $M \subseteq \mathcal{O}_A^{(1)}$ and a subset $S \subseteq \mathcal{O}_A$, if the following conditions* (i) *and* (ii) *hold then M is a centralizing monoid.*

(i) *For any $f \in M$ and any $u \in S$, f and u commute, i.e., $f \perp u$.*

(ii) *For any $g \in \mathcal{O}_A^{(1)} \setminus M$ there exists $w \in S$ such that g does not commute with w, i.e., $g \not\perp w$.*

A subset S in the lemma will be called a *witness* for a centralizing monoid M. We denote by $M(S)$ the centralizing monoid M with S as its witness, i.e., $M(S) = S^* \cap \mathcal{O}_A^{(1)}$. In particular, when $S = \{f\}$ we write $M(f)$ instead of $M(\{f\})$.

Proposition 2.3 *([MR11]) Every centralizing monoid has a finite subset of $\mathcal{O}_A$ as its witness.*

A centralizing monoid M is *maximal* if there is no centralizing monoid M' satisfying $M \subset M' \subset \mathcal{O}_A^{(1)}$. (Here $\subset$ denotes the proper inclusion.)

A function f $(\in \mathcal{O}_A)$ is called a *minimal function* if (i) f generates a minimal clone C and (ii) f has the minimum arity among functions generating C.

Proposition 2.4 *Every maximal centralizing monoid has a singleton set as its witness. Moreover, for every maximal centralizing monoid M there exists a minimal function f $(\in \mathcal{O}_A)$ which serves as a witness of M, i.e., $M = M(f)$.*

3 On a Three-Element Set

In the sequel, we consider only the case where the base set A is $E_3 = \{0, 1, 2\}$. Following [La06], each unary function on E_3 will be denoted as in Table 1.

3.1 Review on Maximal Centralizing Monoids on E_3

Proposition 2.4 asserts that all maximal centralizing monoids can be obtained via minimal functions. Due to B. Csákány ([Cs83]) all minimal clones on E_3 are known. There are 84 minimal clones on E_3. We know all maximal centralizing monoids on E_3 from [MR11].

Proposition 3.1 *([MR11]) On E_3, there are 10 maximal centralizing monoids. Among them,* 3 *maximal centralizing monoids have unary constant functions as their witnesses, and* 7 *maximal centralizing monoids have ternary majority functions which generate minimal clones as their witnesses.*

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5
0	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1
1	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2
2	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2

	s_1	s_2	s_3	s_4	s_5	s_6
0	0	0	1	1	2	2
1	1	2	0	2	0	1
2	2	1	2	0	1	0

	c_0	c_1	c_2
0	0	1	2
1	0	1	2
2	0	1	2

Table 1: Unary Functions in $\mathcal{O}_3^{(1)}$

Clearly, every constant function c_i, taking value i, $(i \in E_3)$ generates a minimal clone. It is known ([Cs83]) that there are 7 minimal clones on E_3 generated by ternary majority functions. Hence, interestingly, every ternary majority function generating a minimal clone serves as a witness of a maximal centralizing monoid.

The following is the set of ternary majority functions generating minimal clones. (The numbering of majority functions is borrowed from [Cs83].) As noted above, each of the following majority functions serves as a witness of some maximal centralizing monoid.

Majority functions generating minimal clones (showing values only for mutually distinct x, y and z):

$$
\begin{aligned}
m_0\,(x,y,z) &= 0 \quad \text{if } |\{x,y,z\}| = 3 \\
m_{364}(x,y,z) &= 1 \quad \text{if } |\{x,y,z\}| = 3 \\
m_{728}(x,y,z) &= 2 \quad \text{if } |\{x,y,z\}| = 3 \\
m_{624}(x,y,z) &= y \quad \text{if } |\{x,y,z\}| = 3 \\
m_{109}(x,y,z) &= \begin{cases} 0 & \text{if } (x,y,z) \in \sigma \\ 1 & \text{if } (x,y,z) \in \tau \end{cases} \\
m_{473}(x,y,z) &= \begin{cases} 1 & \text{if } (x,y,z) \in \sigma \\ 2 & \text{if } (x,y,z) \in \tau \end{cases} \\
m_{510}(x,y,z) &= \begin{cases} 2 & \text{if } (x,y,z) \in \sigma \\ 0 & \text{if } (x,y,z) \in \tau \end{cases}
\end{aligned}
$$

Here σ and τ are the sets of triples: $\sigma = \{0,1,2),(1,2,0),(2,0,1)\}$ and $\tau = \{(0,2,1),(1,0,2),(2,1,0)\}$.

3.2 Strategy to Determine All Centralizing Monoids

The goal of this paper is to determine *all* centralizing monoids on $\{0,1,2\}$. The strategy to achieve this goal is the following.

(1) Choose a maximal centralizing monoid M_{max}.

(2) Find all submonoids of M_{max}.

(3) For each submonoid M of M_{max}, decide if M is a centralizing monoid or not.

(4) Repeat the above procedure for all maximal centralizing monoid.

(5) Finally, collect all submonoids, with repetitions deleted, which have been verified to be centralizing monoids.

A key step in the above strategy is, of course, the step (3). We use the following positive and negative tools to carry out the step (3).

<u>POSITIVE TOOLS</u>

In order to verify that a submonoid M is a centralizing monoid:

(P1) Find a subset $S \subseteq \mathcal{O}_3$ which serves as a witness for M. In other words, find a subset $S \subseteq \mathcal{O}_3$ which satisfies $M = M(S)$.

(P2) Find two centralizing submonoids M_1 and M_2 which satisfy $M_1 \cap M_2 = M$.
(Remark: Let S_1 and S_2 be witnesses of M_1 and M_2, respectively. Then, obviously, $S_1 \cup S_2$ is a witness of $M_1 \cap M_2$.)

NEGATIVE TOOLS

In order to verify that a submonoid M is *not* a centralizing monoid:

(N1) Use Kuznetsov Criterion, which will be explained later (3.2.2 (N1)), to construct some specific function $g \in \mathcal{O}_A^{(1)}$ from functions $f_1, \ldots, f_m \in M$. If such function g does not belong to M then M is not a centralizing monoid.

(N2) Find a submonoid M' which is known to be a centralizing monoid and a submonoid N which is known *not* to be a centralizing monoid. If M' and N satisfy $M \cap M' = N$ then M is not a centralizing monoid.

3.2.1 Witness (P1)

From the witness lemma, $M\ (= M(S)) = \{\, f \in \mathcal{O}_3^{(1)} \mid \forall g \in S,\ f \perp g\}$ for any subset $S \subseteq \mathcal{O}_3$ is a centralizing monoid with S as its witness. There are ample number of examples.

Example 1-1 Let $b_0(x, y)$ and $b_8(x, y)$ be binary functions given by the following tables:

$b_0 =$

$x\backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$b_8 =$

$x\backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	2	2	2

(In passing, we note that b_0 and b_8 are minimal functions.) Then, we see that both

$$M(b_0) = \{j_1, j_2, u_1, u_2, s_1, s_2, c_0, c_1, c_2\} \quad \text{and} \quad M(b_8) = \{j_1, u_2, u_3, s_1, c_0, c_1, c_2\}$$

are centralizing monoids.

Example 1-2 Let $g(x, y)$ be defined by

$$g(x,y) \;=\; \begin{cases} x & \text{if } x+y \neq 3 \\ y & \text{if } x+y = 3\,. \end{cases}$$

Cayley table of g is

$g =$

$x\backslash y$	0	1	2
0	0	0	0
1	1	1	2
2	2	1	2

Then, we see that

$$M(g) \;=\; \{j_0, j_5, u_0, u_5, s_1, s_2, c_0, c_1, c_2\}$$

is a centralizing monoid.

3.2.2 Intersection (P2)

Example 2 Consider three submonoids of $\mathcal{O}_3^{(1)}$:

$$\begin{aligned} M_1 &= \{ j_0, j_5, v_0, v_5, s_1, c_0, c_1, c_2 \} \\ M_2 &= \{ j_5, u_5, v_5, s_1, c_0, c_1, c_2 \} \\ M &= \{ j_5, v_5, s_1, c_0, c_1, c_2 \} \end{aligned}$$

Suppose that it has already been verified that both M_1 and M_2 are centralizing monoids. Then, since $M = M_1 \cap M_2$, it follows that M is also a centralizing monoid.

3.2.3 Kuznetsov Criterion (N1)

For $F \subseteq \mathcal{O}_k$ and p-ary relation ρ on E_k, ρ is *equational* over F if there exist $q \geq p$ and a system Σ of equations over F with variables $x_1, \ldots, x_q$ such that

$$\rho = \{(a_1, \ldots, a_p) \mid (\exists\, a_{p+1}) \ldots (\exists\, a_q)\ (a_1, \ldots, a_q) \text{ is a solution of } \Sigma\}.$$

Moreover, for $f \in \mathcal{O}_k$ and $F \subseteq \mathcal{O}_k$, f is *p-expressible* by F if the graph $f^{\square}$ is equational over F.

Theorem 3.2 *(Kuznetsov criterion)*
For $f \in \mathcal{O}_k$ and $F \subseteq \mathcal{O}_k$, f is p-expressible by F if and only if $f \in F^{**}$.

We can make use of this theorem to verify that some monoid is not a centralizing monoid.

Example 3-1 Take unary functions j_0 and j_2 from Table 1. Consider two equations

$$j_0(x) = j_2(y) \qquad \text{and} \qquad j_2(x) = j_0(y)$$

on variables x, y. The sets of solutions for these equations are $\{ (x,y) \mid j_0(x) = j_2(y) \} = \{ (0,2), (1,0), (1,1), (2,0), (2,1) \}$ and $\{ (x,y) \mid j_2(x) = j_0(y) \} = \{ (0,1), (0,2), (1,1), (1,2), (2,0) \}$, respectively. Hence the set of solutions for the system of these equations is:

$$\{ (x,y) \mid j_0(x) = j_2(y),\, j_2(x) = j_0(y) \} \;=\; \{ (0,2), (1,1), (2,0) \}$$

Since $\{ (0,2), (1,1), (2,0) \}$ is the graph of s_6, where s_6 is the permutation $(0\,2)$, Kuznetsov Criterion asserts that

$$s_6 \in \{ j_0, j_2 \}^{**}.$$

Hence, we conclude that if $j_0, j_2 \in M$ and $s_6 \notin M$ for any submonoid M, then M is not a centralizing monoid.

Example 3-2 In some cases, constant functions are useful to produce non-constant functions. Consider equations

$$j_1(x) = j_5(y) \qquad \text{and} \qquad c_0(x) = j_1(y)$$

where c_0 is the constant function taking value 0. The set of solutions for the system of these equations is:

$$\{ (x,y) \mid j_1(x) = j_5(y),\, c_0(x) = j_1(y) \} \;=\; \{ (0,0), (1,2), (2,0) \}$$

From the fact that

$$\{ (0,0), (1,2), (2,0) \} \;=\; u_1^{\,\square}$$

where u_1 is a two-valued function in Table 1, we claim by Kuznetsov Criterion that

$$u_1 \in \{j_1, j_5, c_0\}^{**}.$$

Hence, for example,

$$M = \{j_1, j_5, s_1, c_0\}$$

is not a centralizing monoid.

Example 3-3 Extra variables bound by the existential quantifier are also helpful. Consider the following:

$$(\exists z)\ s_1(y) = j_1(z)$$

This is equivalent to saying that

$$y \in \{0, 1\}.$$

On the other hand, we have

$$\{(x, y) \mid u_1(x) = u_4(y)\} = \{(0,1), (1,0), (1,2), (2,1)\}.$$

So, we obtain

$$\{(x, y) \mid (\exists z)[u_1(x) = u_4(y) \wedge s_1(y) = j_1(z)]\} = \{(0,1), (1,0), (2,1)\} = j_4{}^{\square}$$

It follows from Kuznetsov Criterion that

$$j_4 \in \{j_1, u_1, u_4, s_1\}^{**}.$$

Hence, for example,

$$M = \{j_1, u_1, u_4, v_4, s_1, c_0, c_1, c_2\}$$

is not a centralizing monoid.

The following is an example of more general results obtained from Kuznetsov Criterion ([MR09]).

Lemma 3.3 *Let M be a submonoid of $\mathcal{O}_A^{(1)}$. Suppose that there exists a non-empty subset N of M such that the intersection of the set of the fixed points of f for all $f \in N$ is a singleton set $\{a\}$ for some $a \in A$. Then the constant function $c_a \in \mathcal{O}_A^{(1)}$ taking value a belongs to M^{**}.*

Example 3-4 An immediate consequence of Lemma 3.3 is, for example, that if M contains $s_2\ (= (1\,2))$ then the constant function c_0 must belong to M^{**}.

3.2.4 Intersection (N2)

It is Immediate to see the following: For monoids M, M', $N \subseteq \mathcal{O}_3^{(1)}$, if M' is a centralizing monoid, N is not a centralizing monoid and $N = M \cap M'$ then M is not a centralizing monoid.

Example 4 Take the following submonoids of $\mathcal{O}_3^{(1)}$:

$$\begin{aligned} M &= \{j_5, u_5, s_1, s_2\} \\ M' &= \{s_1, s_2, c_0\} \\ N &= \{s_1, s_2\} \end{aligned}$$

It is easily verified that M' is a centralizing monoid (because $M' = M(s_2)$), and N is not a centralizing monoid (because $c_0 \in N^{**}$, due to Example 3.4). Then, the equality $N = M \cap M'$ implies that M is not a centralizing monoid.

3.3 Main Result

As stated above, there are 10 *maximal* centralizing monoids on E_3. They are divided into four conjugate classes. (Two submonoids are *conjugate* to each other if one can be obtained from the other by renaming of the elements in the base set E_3.) Each of three conjugate classes consists of 3 maximal centralizing monoids and one conjugate class consists of 1 maximal centralizing monoid. Namely, four conjugate classes are $\{M(c_t) \mid t = 0, 1, 2\}$, $\{M(m_i) \mid i = 0, 364, 728\}$, $\{M(m_j) \mid j = 109, 473, 510\}$ and $\{M(m_{624})\}$ in the notation from Subsection 3.1.

We start from choosing one maximal centralizing monoid M_{max} from each conjugate class and do the following:

Find all submonoids of M_{max}. For each submonoid M of M_{max}, use a positive tool or a negative tool described in Subsection 3.2, to decide if M is a centralizing monoid or not. For each centralizing monoid M, obtain all centralizing monoids which are conjugate to M.

Repeat the above procedure for every representative M_{max} of every conjugate class of maximal centralizing monoids.

Finally, collect all centralizing monoids and delete the repetitions.

Proposition 3.4 (1) *The number of centralizing monoids on* E_3 *is* 192. *There is no centralizing monoid consisting of* k *elements for* $12 \leq k \leq 16$ *and* $18 \leq k \leq 26$.

(2) *The number of the conjugate classes of the centralizing monoids is 48.*

The list of all centralizing monoids on E_3 is given in Tables 2–7. In the table, each row corresponds to a centralizing monoid M and each column corresponds to a unary function f where $\circ$ designates the membership of f in M.

References

[Cs83] Csákány, B, All minimal clones on the three element set, *Acta Cybernet.*, **6**, 1983, 227-238.

[Da79] Danil'tchenko, A. F., On parametrical expressibility of the functions of k-valued logic, *Colloquia Mathematica Societatis János Bolyai*, **28**, Finite Algebra and Multiple-Valued Logic, 1979, 147-159.

[La84] Lau, D., Die Unterhalbgruppen von $(P_3^1; *)$, *Rostock Math. Colloq.*, **26**, 1984, 55-62.

[La06] Lau, D., Function Algebras on Finite Sets, *Springer Monographs in Mathematics*, Springer, 2006.

[MR04] Machida, H. and Rosenberg, I. G., On centralizers of monoids, *Novi Sad Journal of Mathematics*, Vol. **34**, No. **2**, 2004, 153-166.

[MR09] Machida, H. and Rosenberg, I. G., On endoprimal monoids in clone theory, *Proc. 39th Int. Symp. Multiple-Valued Logic*, IEEE, 2009, 167-172.

[MR10] Machida, H. and Rosenberg, I. G., Endoprimal monoids and witness lemma in clone theory, *Proceedings 40th International Symposium on Multiple-Valued Logic*, IEEE, 2010, 195-200.

[MR11] Machida, H. and Rosenberg, I. G., Maximal centralizing monoids and their relation to minimal clones, *Proceedings 41st International Symposium on Multiple-Valued Logic*, IEEE, 2011, 153-159.

[Sza85] Szabó, L. (1985). Characterization of clones acting bicentrally and containing a primitive group, *Acta. Cybernet.*, **7**, 137-142.

List of Centralizing Monoids on E_3

All centralizing monoids on E_3 are listed in the following 5 tables.

- There are 28 columns in each table. Each column, except the first one, corresponds to a unary function on E_3. An item in the first column indicates a "label (name)" and a property (stated below) of a centralizing monoid.
- Each row indicates a centralizing monoid consisting of those unary functions marked ∘ .
- An item in the first column has the form "m-n t_1" or "m-n t_1, t_2". Denote by M the corresponding centralizing monoid. Then, m is the size $|M|$ of M, n means that M is the n-th centralizing monoid in the table among monoids with the same size m, and t_1, resp. t_2, shows that M is conjugate to the first centralizing monoid in the group by the permutation s_{t_1}, resp. s_{t_2}. For example, the monoid labeled "8-1 3" is conjugate to the monoid labeled "8-1 1" by the permutation s_3 (= (0 1)). Also, the monoid labeled "17-1 2, 4" is conjugate to the monoid labeled "17-1 1, 3" by the permutation s_2 (= (1 2)) as well as by the permutation s_4 (= (0 1 2)).

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
27-1	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘	∘
17-1 1,3	∘	∘			∘	∘	∘	∘			∘	∘	∘	∘			∘	∘	∘		∘				∘	∘	∘
17-1 2,4	∘		∘	∘		∘	∘		∘	∘		∘	∘		∘	∘		∘	∘					∘	∘	∘	∘
17-1 5,6		∘	∘	∘	∘			∘	∘	∘	∘			∘	∘	∘	∘		∘	∘					∘	∘	∘
11-1 1,3			∘	∘					∘	∘					∘	∘			∘		∘				∘	∘	∘
11-1 2,4		∘			∘			∘			∘			∘			∘		∘					∘	∘	∘	∘
11-1 5,6	∘					∘	∘					∘	∘					∘	∘	∘					∘	∘	∘
10-1 1,3			∘	∘					∘	∘					∘	∘			∘						∘	∘	∘
10-1 2,4		∘			∘			∘			∘			∘			∘		∘						∘	∘	∘
10-1 5,6	∘					∘	∘					∘	∘					∘	∘						∘	∘	∘
10-2 1,3		∘				∘		∘				∘	∘				∘		∘						∘	∘	∘
10-2 2,4			∘			∘			∘			∘			∘			∘	∘						∘	∘	∘
10-2 5,6		∘		∘					∘		∘				∘		∘		∘						∘	∘	∘
9-1 1,3									∘	∘					∘	∘			∘		∘				∘	∘	∘
9-1 2,4		∘			∘									∘			∘		∘					∘	∘	∘	∘
9-1 5,6	∘					∘	∘					∘							∘	∘					∘	∘	∘
9-2 1,3											∘	∘					∘	∘	∘		∘				∘	∘	∘
9-2 2,4				∘		∘							∘		∘				∘					∘	∘	∘	∘
9-2 5,6		∘	∘					∘	∘										∘	∘					∘	∘	∘
9-3 1,3									∘		∘	∘			∘		∘	∘	∘		∘						∘
9-3 2,4		∘		∘		∘							∘		∘		∘		∘					∘		∘	
9-3 5,6		∘	∘			∘		∘	∘			∘							∘	∘					∘		
9-4																			∘	∘	∘	∘	∘	∘	∘	∘	∘

Table 2: Centralizing Monoids on E_3 (Size: 27–9)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
8-1 1			○	○					○	○									○						○	○	○
8-1 3			○	○											○	○			○						○	○	○
8-1 4								○			○			○			○		○						○	○	○
8-1 2		○			○			○			○								○						○	○	○
8-1 5	○					○							○					○	○						○	○	○
8-1 6							○					○	○					○	○						○	○	○
8-2 1,3									○	○					○	○			○						○	○	○
8-2 2,4		○			○									○			○		○						○	○	○
8-2 5,6	○					○	○					○							○						○	○	○
7-1 1												○		○			○		○						○	○	○
7-1 3							○					○					○		○						○	○	○
7-1 4	○					○									○				○						○	○	○
7-1 2						○									○	○			○						○	○	○
7-1 5		○							○	○									○						○	○	○
7-1 6		○			○				○										○						○	○	○
7-2 1		○						○									○		○						○	○	○
7-2 3						○						○	○						○						○	○	○
7-2 4						○						○						○	○						○	○	○
7-2 2			○						○						○				○						○	○	○
7-2 5				○					○						○				○						○	○	○
7-2 6		○									○						○		○						○	○	○
7-3 1,3									○						○				○		○				○	○	○
7-3 2,4		○															○		○					○	○	○	○
7-3 5,6						○						○							○	○					○	○	○
7-4 1,3			○	○															○		○				○	○	○
7-4 2,4								○			○								○					○	○	○	○
7-4 5,6													○					○	○	○					○	○	○
7-5 1,3											○	○					○	○	○		○						○
7-5 2,4				○		○							○		○				○					○		○	
7-5 5,6		○	○					○	○										○	○					○		
6-1 1														○			○		○						○	○	○
6-1 3							○					○							○						○	○	○
6-1 4	○					○													○						○	○	○
6-1 2															○	○			○						○	○	○
6-1 5									○	○									○						○	○	○
6-1 6		○			○														○						○	○	○
6-2 1								○									○		○						○	○	○
6-2 3												○	○						○						○	○	○
6-2 4						○												○	○						○	○	○
6-2 2			○												○				○						○	○	○
6-2 5				○					○										○						○	○	○
6-2 6		○									○								○						○	○	○

Table 3: Centralizing Monoids on E_3 (Size: 8–6)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
6-3 1		○						○											○						○	○	○
6-3 3						○							○						○						○	○	○
6-3 4												○						○	○						○	○	○
6-3 2			○						○										○						○	○	○
6-3 5				○											○				○						○	○	○
6-3 6											○						○		○						○	○	○
6-4 1												○		○			○		○							○	○
6-4 3							○					○					○		○						○		○
6-4 4	○					○									○				○						○	○	
6-4 2						○									○	○			○							○	○
6-4 5		○							○	○									○						○		○
6-4 6		○			○				○										○						○	○	
6-5 1		○				○		○				○							○						○		
6-5 3		○				○							○				○		○							○	
6-5 4									○			○			○			○	○								○
6-5 2			○			○			○			○							○						○		
6-5 5		○		○											○		○		○							○	
6-5 6									○		○				○		○		○								○
6-6 1,3									○						○				○						○	○	○
6-6 2,4		○															○		○						○	○	○
6-6 5,6						○						○							○						○	○	○
6-7 1,3												○					○		○						○	○	○
6-7 2,4						○									○				○						○	○	○
6-7 5,6		○							○										○						○	○	○
6-8 1,3			○	○															○						○	○	○
6-8 2,4								○			○								○						○	○	○
6-8 5,6													○					○	○						○	○	○
6-9																			○			○	○		○	○	○
5-1 1																	○		○						○	○	○
5-1 3												○							○						○	○	○
5-1 4						○													○						○	○	○
5-1 2															○				○						○	○	○
5-1 5									○										○						○	○	○
5-1 6		○																	○						○	○	○
5-2 1								○											○						○	○	○
5-2 3													○						○						○	○	○
5-2 4																		○	○						○	○	○
5-2 2			○																○						○	○	○
5-2 5				○															○						○	○	○
5-2 6											○								○						○	○	○

Table 4: Centralizing Monoids on E_3 (Size: 6–5)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
5-3 1														○			○		○							○	○
5-3 3							○					○							○						○		○
5-3 4	○					○													○						○	○	
5-3 2															○	○			○							○	○
5-3 5									○	○									○						○		○
5-3 6		○			○														○						○	○	
5-4 1												○					○		○							○	○
5-4 3												○					○		○						○		○
5-4 4						○									○				○						○	○	
5-4 2						○									○				○							○	○
5-4 5		○							○										○						○		○
5-4 6		○							○										○						○	○	
5-5 1,3																			○		○				○	○	○
5-5 2,4																			○					○	○	○	○
5-5 5,6																			○	○					○	○	○
5-6 1,3									○						○				○		○						○
5-6 2,4		○															○		○					○		○	
5-6 5,6						○						○							○	○					○		
4-1 1																	○		○							○	○
4-1 3												○							○						○		○
4-1 4						○													○						○	○	
4-1 2															○				○							○	○
4-1 5									○										○						○		○
4-1 6		○																	○						○	○	
4-2 1												○							○							○	○
4-2 3																	○		○						○		○
4-2 4															○				○						○	○	
4-2 2						○													○							○	○
4-2 5		○																	○						○		○
4-2 6									○										○						○	○	
4-3 1		○						○											○						○		
4-3 3						○							○						○							○	
4-3 4												○						○	○								○
4-3 2			○						○										○						○		
4-3 5				○											○				○							○	
4-3 6											○						○		○								○
4-4 1,3									○						○				○								○
4-4 2,4		○															○		○							○	
4-4 5,6						○						○							○						○		
4-5 1,3												○					○		○								○
4-5 2,4						○									○				○							○	
4-5 5,6		○							○										○						○		
4-6																			○						○	○	○

Table 5: Centralizing Monoids on E_3 (Size: 5–4)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
3-1 1		○																	○						○		
3-1 3						○													○							○	
3-1 4												○							○								○
3-1 2									○										○						○		
3-1 5															○				○							○	
3-1 6																	○		○								○
3-2 1						○													○						○		
3-2 3		○																	○							○	
3-2 4									○										○								○
3-2 2												○							○						○		
3-2 5																	○		○							○	
3-2 6															○				○								○
3-3 1			○																○						○		
3-3 3				○															○							○	
3-3 4											○								○								○
3-3 2								○											○						○		
3-3 5													○						○							○	
3-3 6																		○	○								○
3-4 1														○			○		○								
3-4 3							○					○							○								
3-4 4	○					○													○								
3-4 2															○	○			○								
3-4 5									○	○									○								
3-4 6		○			○														○								
3-5 1,3		○				○													○								
3-5 2,4									○			○							○								
3-5 5,6															○		○		○								
3-6 1,3																			○						○	○	
3-6 2,4																			○						○		○
3-6 5,6																			○							○	○
3-7 1,3																			○		○						○
3-7 2,4																			○					○		○	
3-7 5,6																			○	○					○		
3-8																			○			○	○				
2-1 1		○																	○								
2-1 3						○													○								
2-1 4												○							○								
2-1 2									○										○								
2-1 5															○				○								
2-1 6																	○		○								
2-2 1,3																			○								○
2-2 2,4																			○							○	
2-2 5,6																			○						○		
1-1																			○								

Table 6: Centralizing Monoids on E_3 (Size: 3–1)